Published by Institute of Physics Publishing for SISSA
Received: *February 22, 2005*
Accepted: *March 2, 2005*
Published: *April 15, 2005*

Spinning strings in $AdS_5 \times S^5$: one-loop correction to energy in $\mathrm{SL}(2)$ sector

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Abstract: We consider a circular string with spin S in AdS_5 wrapped around big circle of S^5 and carrying also momentum J. The corresponding $N = 4$ SYM operator belongs to the SL(2) sector, i.e. has $\mathrm{tr}(D^S Z^J) + \cdots$ structure. The leading large J term in its 1-loop anomalous dimension can be computed using Bethe ansatz for the SL(2) spin chain and was previously found to match the leading term in the classical string energy. The string solution is stable at large J, and the lagrangian for string fluctuations has constant coefficients, so that the 1-loop string correction to the energy E_1 is given simply by the sum of characteristic frequencies. Curiously, we find that the leading term in the zero-mode part of E_1 is the same as a $1/J$ correction to the one-loop anomalous dimension on the gauge theory (spin chain) side that was found in [1]. However, the contribution of non-zero string modes does not vanish. We also discuss the "fast string" expansion of the classical string action which coincides with the coherent state action of the SL(2) spin chain at the first order in λ, and extend this expansion to higher orders clarifying the role of the S^5 winding number.

Keywords: Superstrings and Heterotic Strings, AdS-CFT and dS-CFT Correspondence.

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Contents

1. Introduction

Study of semiclassical rotating strings in $AdS_5 \times S^5$ which extended earlier work [2, 3] to cases with several large angular momenta have led to interesting developments in understanding and checking the AdS/CFT duality. The classical energy of such strings has "regular" expansion [4, 5] in the effective coupling $\tilde{\lambda} = \frac{\lambda}{J^2}$, i.e. $E = J(1 + c_1 \frac{\lambda}{J^2} + c_2 \frac{\lambda^2}{J^4} + \cdots)$, where J is total angular momentum and λ is the square of string tension or the 't Hooft coupling on the SYM side. This prompted a possibility [5] of direct comparison with perturbative anomalous dimensions of the corresponding single-trace operators in $\mathcal{N} = 4$ SYM theory.

Indeed, the precise agreement was established at the first two leading orders in λ [6]–[10] (for reviews and further references see [11]–[15]). This agreement was found to break down (as in the near-BMN [4, 16] case [17]) at the next λ^3 order [9, 10]. A natural explanation is an order-of-limits problem [9] and [18]–[20]: on the string side one first takes J to be large to suppress quantum (α') string corrections and then expands E/J in $\tilde{\lambda} = \frac{\lambda}{J^2}$ (so that λ is effectively large), while on the SYM side one uses perturbation theory in λ and then expands each loop correction to anomalous dimensions in $1/J$. The exact agreement at the first two leading orders appears to be due to a special structure of the one- and two-loop dilatation operator in $\mathcal{N} = 4$ SYM theory [21]–[23] and should be essentially a consequence of large supersymmetry of the theory.

To understand how one may still preserve the equality between the exact string energies $E(J,\lambda)$ and the exact SYM anomalous dimensions $\Delta(\lambda, J)$, i.e. to explain the interpolation between the perturbative string and the perturbative SYM expressions it is important to go beyond the classical string theory and compare the 1-loop corrections to string energies to subleading in $1/J$ terms in the SYM anomalous dimensions. This is also important in order try to provide more data for checking a non-perturbative Bethe ansatz proposal of [19].

On the string side, the computation of 1-loop corrections to classical string energies is relatively straightforward for a special class of "homogeneous" circular strings [5, 24] for which the fluctuation lagrangian has constant coefficients [25, 24]. On the SYM side, finding the corresponding subleading $1/J^2$ terms in the one-loop anomalous dimensions amounts to computing corrections to the thermodynamic limit of the Bethe ansatz equations and was previously done in the SU(2) sector [26] only for a special state corresponding to a circular string rotating in S^5 with two equal angular momenta [5]. The latter solution is, however, unstable [5, 25], so that the 1-loop correction to its energy contains formally an imaginary part. Then a mismatch between its real part [27] and a real SYM expression of [26] may be viewed as an inconclusive evidence of a disagreement.

Here we shall perform a 1-loop string computation very similar to the one carried out in [25, 27] but for a *stable* circular 2-spin string solution found in [24]. In conformal gauge, it is represented by a string of fixed radius (wound k times) lying on a plane in AdS_5 and rotating along itself so that it carries one component S of spin in AdS_5. The string is also wound (m times) around a big circle of S^5 and is rotating along itself with momentum J (the "level matching" constraint implies that $kS + mJ = 0$). The fast enough rotation stabilizes this solution. This configuration may be visualized as a circular spiral.[1] It is an $AdS_5 \times S^5$ analog of a closed string in flat $\mathbb{R}_t \times \mathbb{R}^2 \times S^1$ space-time which is wrapped k times on a constant-radius circle in $\mathbb{R}^2$ and also m times on S^1 and rotating along itself in each circle (alternatively, the flat-space solution may be viewed as a left-moving wave along a circular string wrapped on S^1 and it thus represents a BPS state; the $AdS_5 \times S^5$ solution is no longer BPS).

The classical energy of this (S, J) string solution has the following expansion at large J with fixed $\frac{S}{J}$ [24]:

$$E_0 = J + S + \frac{\lambda k^2}{2J}\frac{S}{J}\left(1 + \frac{S}{J}\right) + O\left(\frac{\lambda^2}{J^3}\right). \tag{1.1}$$

The corresponding SYM operator belongs to the closed $\mathrm{SL}(2,\mathbb{R})$ sector in which the 1-loop SYM dilatation operator is the same as the hamiltonian of the SL(2) Heisenberg spin chain with length $L = J$ [28], i.e. it has the structure $\mathrm{tr}(D^S Z^J) + \cdots$, where one is to sum over orderings of D's and Z's to get an eigen-state of the dilatation operator ($D = D_1 + iD_2$ is a covariant derivative in $\mathbb{R}^4$ and $Z = \phi_1 + i\phi_2$ is a chiral combination of SYM scalars). The analysis of the thermodynamic limit of the 1-loop SL(2) spin chain Bethe ansatz equations showed [1] that there is indeed a SYM state whose 1-loop anomalous dimension $\Delta(S, J; \lambda)$

[1] This is a fixed time profile of the string, so a more appropriate name would be a "spiral string".

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matches precisely the order λ term in (1.1). A subleading $1/J^2$ term in the 1-loop part of Δ can be found [1] as in the SU(2) sector [26] from the correction to the thermodinamic limit of the Bethe ansatz equations. It turned out to be given simply by the leading-order term multiplied by $-1/J$, i.e. [1]

$$\Delta = \lambda \frac{k^2}{2} \frac{S}{J} \left(1 + \frac{S}{J}\right) \left[\frac{1}{J} - \frac{1}{J^2} + O\left(\frac{1}{J^3}\right)\right] + O(\lambda^2)\,. \tag{1.2}$$

Our aim below will be to compute the 1-loop string correction E_1 to the energy (1.1) of the circular (S, J) string solution. We shall find that, as expected [25], the 1-loop correction is suppressed compared to the classical expression (1.1) by an extra power of J, and will compare the coefficient of the order λ/J^2 term in E_1 to the coefficient of the $\frac{\lambda}{J^2}$ term on the SYM side (1.2).

As in the SU(2) case considered in [27], we shall conclude (in section 4 below) that there is an apparent disagreement between the subleading in $1/J$ string result of this paper and the spin chain result of [26, 1]. A curious observation is that the agreement would hold if we were to keep only the contributions of the zero modes of the string fluctuations, i.e. to consider a "homogeneous" approximation in which the string fluctuations depend only on τ but not on σ.

One may think, of course, that there is no a priori reason to expect the exact agreement, and, as in [9, 18, 27], any disagreement should be attributed to the different order of limits taken on the gauge theory and the string theory sides of the AdS/CFT duality.[2] In particular, the spin chain Bethe ansatz and the string theory computations of the subleading $1/J$ corrections are not directly related: while the Bethe ansatz approach does reproduce part of the spectrum of the bosonic string fluctuations near a given semiclassical solution [6, 1],[3] the string 1-loop computation involves summing over *all* bosonic *and* fermionic modes.[4]

Before turning to the computation of the one-loop correction to the energy of the circular solution we shall start (in section 2) with a discussion of the "fast string" limit of the classical string action in the SL(2) or (S, J) sector. The comparison of the reduced

[2]There may be potential subtleties related to how one takes limits in the (λ, J) parameter space [18]. In the cases of subleading $1/J^2$ corrections to the BMN point-like state and the $1/J$ corrections to the spinning strings the disagreements could, in principle, start already at order λ [27, 20]. Still, one could hope, by analogy with what was found for the large J correspondence, that disagreements should start only at order λ^3 (e.g., due to a specific structure of the 3-loop SYM dilatation operator).

[3]There are two type of string fluctuations near circular spinning strings (see [5, 25] and below): whose frequencies scale at large $\mathcal{J} = \frac{J}{\sqrt{\lambda}}$ as (i) $\omega^{(1)} \sim \mathcal{J} + \frac{a}{\mathcal{J}}$ and as (ii) $\omega^{(2)} \sim \frac{b}{\mathcal{J}}$ (the corresponding energies are $\delta E \sim \frac{\omega}{\mathcal{J}}$). Fermionic frequencies belong to the first type. The first ("BMN") type of fluctuations correspond to deformations away from a particular (SU(2) or SL(2)) sector, and are not "seen" directly in the Bethe ansatz equations for the corresponding spin chain. The second type of fluctuations correspond to deformations within a given sector and are captured by the spin chain [6, 29, 10] and the corresponding coherent state ("Landau-Lifshitz") action [30, 31, 32].

[4]In the Bethe ansatz approach fluctuations near a given semiclassical state described by a smooth Bethe root distribution are obtained by pulling one Bethe root out of a continuous distribution [6]. Summing over fluctuations would look like averaging over a set of Bethe states, which is not what was done in [26] to compute the $1/J$ correction. We are grateful to K. Zarembo for this remark.

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sigma model appearing in the "fast string" limit of the string action to the coherent-state ("Landau-Lifshitz") action on the ferromagnetic spin chain side provided a simple and universal way of matching the string and gauge theory semiclassical states as well as near-by fluctuations [30]–[38].[5] In the SL(2) sector that was done in [33] and also in [36]. Below we will extend the derivation of the string action limit in [33] to the next order of expansion in $\frac{\lambda}{J^2}$ and will also include the general case of string configurations with non-zero winding number m in the $S^1 \subset S^5$ direction. The spin-chain coherent state action cannot depend on m, and, indeed, m does not appear also in the relevant limit of the string sigma model action; instead, it classifies particular solutions of the resulting "Landau-Lifshitz" equations. In particular, we shall explain (in section 3) how the circular (S,J) solution of [24] which has $m \neq 0$ can be obtained from the SL(2) "Landau-Lifshitz" equations (order by order in $\frac{\sqrt{\lambda}}{J}$). Closely related aspects of the general $m \neq 0$ sigma model solutions in the SL(2) sector and their relation to the Bethe root distributions on the spin chain side were discussed in [1].

The circular string state of SL(2) sector discussed in this paper may be of interest also in a more general context of gauge–string duality. Similar solutions exist in any space with $AdS_3 \times S^1$ part, e.g., in $AdS_5 \times S^1$ which is relevant for a description of less supersymmetric 4-d conformal theories [43]. Also, the corresponding operators in the SL(2) sector of $\mathcal{N} = 4$ SYM have direct connection to operators appearing in pure YM or $\mathcal{N} = 1$ SYM theory [44].

We shall start in section 2 with a discussion of large J expansion of the classical string action in the SL(2) sector, then in section 3 review the (S,J) circular string solution, and, finally, in section 4 present the computation of the 1-loop string correction to its energy. In Appendix A, which is a generalization of section 2, we shall discuss the "fast string" limit of the $AdS_5 \times S^5$ sigma model in a more general case of five non-vanishing spins. In Appendix B we shall give some details of computation of fermionic characteristic frequencies used in section 4. In Appendix C we shall discuss the Landau-Lifshitz frequencies for the (S,J) solution.

2. "Fast-string" limit of string action in SL(2) sector

The SYM operators from the SL(2) sector form a closed set under renormalization to all orders in perturbation theory in λ [45, 28, 44]. As in the SU(2) sector [30, 31], here it makes sense to compare the string theory and a coherent-state spin chain lagrangians at large spins to all orders in the coupling $\tilde{\lambda} = \frac{\lambda}{J^2}$. This comparison was initiated at the leading order in [33]. Here using a non-conformal "homogeneous" gauge similar to the one used in the SU(2) case [31, 35] we will show how to compute subleading terms in the expansion of the string action for a general (S,J) closed string configuration with a non-zero winding m along a big circle in S^5 (ref. [33] considered the $m = 0$ case). The two-loop dilatation operator in the SL(2) sector is not known explicitly (yet the corresponding

[5] A complementary approach demonstrating general agreement between the string and gauge theory semiclassical states at the two leading orders in λ is based on directly relating the thermodynamic limit of the Bethe ansatz equations to the integral equation representing the general integrability-based solution of the corresponding sector of the classical $AdS_5 \times S^5$ sigma model [14, 1] and [39]–[42].

Bethe equations and the associated 1-d S-matrix were recently found in [20]), so a direct comparison of the λ^2 subleading term in the string action found below to the 2-loop term in the coherent state "Landau-Lifshitz" action on the spin chain side (along the same lines as in the SU(2) case [31]) is not possible at present.

Let us first set up our notation by recalling the bosonic part of the $AdS_5 \times S^5$ string action (see [5, 11, 24, 33])

$$I = \sqrt{\lambda} \int d\tau \int_0^{2\pi} \frac{d\sigma}{2\pi} \sqrt{-g}(L_S + L_{AdS}) \tag{2.1}$$

$$L_S = -\frac{1}{2} g^{ab} \partial_a X_M \partial_b X_M + \frac{1}{2} \Lambda (X_M X_M - 1) \tag{2.2}$$

$$L_{AdS} = -\frac{1}{2} g^{ab} \eta^{PQ} \partial_a Y_P \partial_b Y_Q - \frac{1}{2} \tilde{\Lambda} (\eta^{PQ} Y_P Y_Q + 1)\,, \tag{2.3}$$

where X_M ($M = 1, \ldots, 6$) and Y_P ($P = 0, \ldots, 5$) are the embedding coordinates of $\mathbb{R}^6$ and of $\mathbb{R}^{2,4}$ (the latter with the metric $\eta_{PQ} = (-1, +1, +1, +1, +1, -1)$). Let us define 3+3 complex coordinates:

$$\mathrm{Y}_0 \equiv Y_5 + iY_0\,, \quad \mathrm{Y}_1 \equiv Y_1 + iY_2\,, \quad \mathrm{Y}_2 \equiv Y_3 + iY_4\,, \quad \mathrm{Y}_r^* \mathrm{Y}^r = -1\,, \tag{2.4}$$

$$\mathrm{X}_1 \equiv X_1 + iX_2\,, \quad \mathrm{X}_2 \equiv X_3 + iX_4\,, \quad \mathrm{X}_3 \equiv X_5 + iX_6\,, \quad \mathrm{X}_i^* \mathrm{X}_i = 1\,. \tag{2.5}$$

Then the $AdS_5 \times S^5$ metric is $ds^2 = d\mathrm{Y}_r^* d\mathrm{Y}^r + d\mathrm{X}_i^* d\mathrm{X}_i$ where $r, s = 0, 1, 2$ and $i, j, k = 1, 2, 3$ ($\mathrm{Y}^r = \eta^{rs} \mathrm{Y}_s$, with $\eta^{rs} = (-1, 1, 1)$). In terms of the usual global time, radial and angular coordinates one has

$$\mathrm{Y}_0 = \cosh\rho\, e^{it}, \quad \mathrm{Y}_1 = \sinh\rho\, \sin\theta\, e^{i\phi_1}, \quad \mathrm{Y}_2 = \sinh\rho\, \cos\theta\, e^{i\phi_2} \tag{2.6}$$

$$\mathrm{X}_1 = \sin\gamma \cos\psi\, e^{i\varphi_1}, \quad \mathrm{X}_2 = \sin\gamma \sin\psi\, e^{i\varphi_2}, \quad \mathrm{X}_3 = \cos\gamma\, e^{i\varphi_3}\,. \tag{2.7}$$

Separating the common phases of Y_r and X_i

$$\mathrm{Y}_r = e^{iy} V_r, \qquad \mathrm{X}_i = e^{i\alpha} U_i\,, \qquad V_r^* V^r = -1, \qquad U_i^* U_i = 1\,, \tag{2.8}$$

the $AdS_5 \times S^5$ metric becomes:

$$ds^2 = -(Dy)^2 + (D\alpha)^2 + D^* V_r^* DV^r + D^* U_i^* DU_i\,, \tag{2.9}$$

where

$$Dy = dy + B\,, \qquad DV_r = dV_r - iBV_r\,, \qquad B = iV_r^* dV^r\,, \tag{2.10}$$

$$D\alpha = d\alpha + C\,, \qquad DU_i = dU_i - iCU_i\,, \qquad C = -iU_i^* dU_i\,. \tag{2.11}$$

In what follows we shall consider the string configurations in the (S, J) sector where the strings may wound around and move along a big circle in S^5 and may also move in $(1, 2)$ plane in AdS_5, i.e.

$$\mathrm{X} \equiv \mathrm{X}_1 = e^{i\alpha}\,, \qquad \mathrm{X}_2 = \mathrm{X}_3 = 0 \tag{2.12}$$

$$\mathrm{Y}_0 = e^{iy} V_0 = \cosh\rho\, e^{it}\,, \qquad \mathrm{Y}_1 = e^{iy} V_1 = \sinh\rho\, e^{i\phi_1}\,, \qquad \mathrm{Y}_2 = 0\,. \tag{2.13}$$

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We may fix the obvious U(1) symmetry of Y_r: $y \to y - \chi$, $V_r \to e^{i\chi} V_r$ (from now on we will assume that $r, s = 0, 1$) by the following choice of y

$$y = t\,, \tag{2.14}$$

so that $V_0 = \cosh\rho$, $V_1 = \sinh\rho\, e^{i(\phi_1 - t)}$. Then the lagrangian in (2.1) reduces to

$$L = -\frac{1}{2}\sqrt{-g}\; g^{ab}\left(-D_a t D_b t + \partial_a\alpha\partial_b\alpha + D_a^* V_r^* D_b V^r\right). \tag{2.15}$$

As in [31] the idea is to assume that the string is moving fast and gauge fix t and the momentum conjugate to the "fast" coordinate α, getting an effective action for the "slow" transverse coordinates V_r only.

The conserved charges corresponding to the translations in t, rotations in AdS_5 or $Y_1 \to e^{i\beta} Y_1$ and translations in α

$$E \equiv \sqrt{\lambda}\;\mathcal{E}, \qquad S \equiv \sqrt{\lambda}\,\mathcal{S}, \qquad J \equiv \sqrt{\lambda}\,\mathcal{J} \tag{2.16}$$

are given by

$$\mathcal{E} - \mathcal{S} = \int_0^{2\pi} \frac{d\sigma}{2\pi}\, p_t, \qquad p_t = -\sqrt{-g}g^{0a} D_a t\,, \tag{2.17}$$

$$\mathcal{S} = -\int_0^{2\pi} \frac{d\sigma}{2\pi}\sqrt{-g}g^{0a}\left[V_1 V_1^* D_a t - \left(\frac{i}{2} V_1^* D_a V_1 + \text{c.c.}\right)\right] \tag{2.18}$$

$$\mathcal{J} = \int_0^{2\pi} \frac{d\sigma}{2\pi}\, p_\alpha\,, \qquad p_\alpha = -\sqrt{-g}g^{0a}\;\partial_a\alpha\,. \tag{2.19}$$

Following [35], we may replace α by the dual coordinate $\tilde{\alpha}$

$$\sqrt{-g}g^{ab}\partial_b\alpha = -\epsilon^{ab}\partial_b\tilde{\alpha}\,. \tag{2.20}$$

Solving for the 2-d metric g_{ab} in (2.15) we get the Nambu-type lagrangian

$$L = -\sqrt{h}\,, \qquad h = |\det h_{ab}|\,, \tag{2.21}$$

$$h_{ab} = -D_a t D_b t + \partial_a\tilde{\alpha}\partial_b\tilde{\alpha} + D_{(a}^* V_r^* D_{b)} V^r\,. \tag{2.22}$$

To fix the world-sheet reparametrization freedom we choose, as in [31, 35], the following two gauge conditions

$$(i)\; t = \tau\,, \qquad (ii)\; \tilde{\alpha} = \mathcal{J}\sigma\,, \quad \text{i.e.} \quad p_\alpha = \mathcal{J} = \text{const}\,. \tag{2.23}$$

The first ensures that the space-time and the world-sheet energies are the same; the second implies that J is distributed homogeneously along the string coordinate σ (which is the case on the spin chain side). Then the string action becomes

$$I = J\int d\tau \int_0^{2\pi} \frac{d\sigma}{2\pi}\tilde{L}\,, \qquad \tilde{L} = -\mathcal{J}^{-1}\sqrt{h}\,, \tag{2.24}$$

with $(B_a = iV_r^*\partial_a V^r)$

$$h = (\mathcal{J}^2 - B_1^2 + |D_1V_r|^2)[|D_0V_r|^2 - (1+B_0)^2] - \left[\frac{1}{2}(D_0V_r^* D_1V^r + c.c.) - B_1(1+B_0)\right]^2 . \tag{2.25}$$

The next step [30, 33, 31] is to expand in large $\mathcal{J}$ assuming that higher powers of time derivatives of V_r are suppressed. To define an expansion in $\frac{1}{\mathcal{J}^2} = \frac{\lambda}{J^2} \equiv \tilde{\lambda}$ it is useful to rescale τ so that the leading order term does not contain $\tilde{\lambda}$: $\tau \to \mathcal{J}^2\tau$, $\partial_0 \to \frac{1}{\mathcal{J}^2}\partial_0$. Then

$$\begin{aligned}\tilde{L} &= -\mathcal{J}^2 - B_0 - \frac{1}{2}|D_1V_r|^2 + \\ &\quad + \frac{1}{8\mathcal{J}^2}\left[|D_1V_r|^4 + 4|D_0V_r|^2 - 4B_0|D_1V_r|^2 + 4B_1(D_0^*V_r^*D_1V^r + \text{c.c.})\right] + O\left(\frac{1}{\mathcal{J}^4}\right)\end{aligned} \tag{2.26}$$

Finally, we may eliminate the time derivative terms by field redefinitions [31] or, to leading order, simply using the "Landau-Lifshitz" equation [33] following from (2.26)[6]

$$i\partial_0 V^r - \frac{1}{2}\partial_1^2 V^r - \partial_1 V^r \partial_1 V^s V_s^* - \frac{1}{2}\partial_1(\partial_1 V^s V_s^*)\ V^r \ + O\left(\frac{1}{\mathcal{J}^2}\right) = 0 . \tag{2.27}$$

We get

$$\begin{aligned}\tilde{L} &= -\mathcal{J}^2 - iV_r^*\partial_0 V^r - \frac{1}{2}|D_1V_r|^2 + \\ &\quad + \frac{1}{8\mathcal{J}^2}\left[4|D_1V_r|^4 + |D_1^2V^r|^2 - 2(V_s^*\partial_1V_s)^2|D_1V_r|^2 - 2(V_s^*\partial_1V_sD_1V^rD_1^2V_r^* + \text{c.c.})\right] .\end{aligned} \tag{2.28}$$

As was shown in [33], the first non-trivial (order $\mathcal{J}^0$) term here matches the coherent-state effective action following from the 1-loop dilatation operator in the SL(2) sector [28, 44] (the hamiltonian of the SL(2) Heisenberg ferromagnetic).

Rescaling back $\tau \to \frac{1}{\mathcal{J}^2}\tau$, we find that the corresponding 2d energy or hamiltonian is (in view of our gauge choice $t=\tau$, is the same as the $E-S$ in (2.19))

$$\begin{aligned}E - S = J &+ \frac{\lambda}{2J}\int_0^{2\pi}\frac{d\sigma}{2\pi}\ |D_1V_r|^2 - \\ &- \frac{\lambda^2}{8J^3}\int_0^{2\pi}\frac{d\sigma}{2\pi}\Big[4|D_1V_r|^4 + |D_1^2V_r|^2 - 2(V_s^*\partial_1V^s)^2|D_1V_r|^2 - \\ &\qquad - 2(V_s^*\partial_1V^sD_1V^rD_1^2V_r^* + \text{c.c.})\Big] + O(\frac{\lambda^3}{J^5}) .\end{aligned} \tag{2.29}$$

The expression for the spin $\mathcal{S}$ in (2.18) expanded in large is

$$\begin{aligned}\mathcal{S} &= \mathcal{J}\int_0^{2\pi}\frac{d\sigma}{2\pi}V_1V_1^* + \\ &\quad + \frac{1}{2\mathcal{J}}\int_0^{2\pi}\frac{d\sigma}{2\pi}\left[V_1V_1^*|D_1V_r|^2 - B_1(iV_1^*D_1V_1 + c.c.) - (iV_1^*D_0V_1 + \text{c.c.})\right] + O\left(\frac{1}{\mathcal{J}^3}\right) ,\end{aligned} \tag{2.30}$$

where the time derivatives should be again eliminated using (2.27).

[6] To obtain this equation one should vary (2.26) taking into account the constraint $V^sV_s^* = -1$ (which may be imposed using, e.g., a Lagrange multiplier).

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In the above discussion α was assumed to be a general function of τ and σ, which, due to the periodicity condition, should be subject to

$$\alpha(\sigma+2\pi,\tau)=\alpha(\sigma,\tau)+2\pi m\,, \qquad \text{i.e.} \qquad \int_0^{2\pi}\frac{d\sigma}{2\pi}\partial_1\alpha=m\,, \tag{2.31}$$

where m is an integer winding number along the S^5 circle. A possible winding along the AdS_5 circle is (by our choice of $y=t=\tau$) absorbed into V_1. Using (2.23),(2.20) we find $m=-\mathcal{J}\int_0^{2\pi}\frac{d\sigma}{2\pi}\sqrt{-g}\,g^{01}$, and after plugging in the expression for the 2-d metric $g_{ab}=h_{ab}$ in (2.22) we conclude that

$$m=-\mathcal{J}\int_0^{2\pi}\frac{d\sigma}{2\pi}\frac{h_{01}}{\sqrt{h}}\,, \qquad h_{01}=-B_1(1+B_0)+\frac{1}{2}(D_0^*V_r^*D_1V^r+\text{c.c.})\,. \tag{2.32}$$

This is an additional constraint which should be imposed on any particular solution of equations following from (2.28). Expanding this condition for large $\mathcal{J}$ and eliminating time derivatives using (2.27) we get

$$m=\int_0^{2\pi}\frac{d\sigma}{2\pi}\left(iV_r^*\partial_1V^r-\frac{1}{2\mathcal{J}^2}\left[iV_s^*\partial_1V^s|D_1V_r|^2+\frac{1}{2}(iD_1V^rD_1^2V_r^*+\text{c.c.})\right]\right)+O\left(\frac{1}{\mathcal{J}^4}\right). \tag{2.33}$$

We conclude that while the S^5 winding number m does not enter the effective lagrangian, it appears in the constraint on its solutions. This is in agreement with the spin-chain side considerations where m enters a constraint on Bethe roots but not the algebraic Bethe equations [1] or the coherent-state effective action [33].

A similar procedure of taking a "fast-string" limit of the string action can be applied in a more general case of 5 non-vanishing spins (S_1,S_2,J_1,J_2,J_3); we shall discuss it in appendix A.[7]

3. Circular string solution in $AdS_3\times S^1\subset AdS_5\times S^5$

In this section we shall review a remarkably simple circular string solution of (2.1) found in [24] representing a particular semiclassical (S,J) state in the SL(2) sector. The string is positioned in a plane in AdS_3 and is also wrapped on S^1 in AdS_5. As we shall explain below, this configuration (its expansion in large $\mathcal{J}$) can be obtained also as a solution of the "Landau-Lifshitz" equations (2.27).

It is useful first to discuss its flat-space analog by considering a closed string moving in $\mathbb{R}^{1,2}\times S^1$ instead of $AdS_3\times S^1$ (then $ds^2=-dt^2+d\rho^2+\rho^2d\phi^2+R^2d\alpha^2$, R is a radius of S^1).[8] Let us use conformal gauge and consider a rigid circular string lying in $\mathbb{R}^2$ (with coordinates Y_1,Y_2) and rotating along itself while being also wound along S^1 and having

[7] Even though the corresponding set of SYM operators is, in general, not closed under renormalisation beyond one loop, it was argued in [10] that non-closed sectors (like SU(3) and SO(6)) may be viewed as closed in the thermodynamic limit even at higher loops. It then makes sense to compare the corresponding anomalous dimensions or semiclassical effective actions they are described by to a limit of string theory action even beyond leading order in the effective coupling $\tilde{\lambda}$.

[8] Similar solution exists also in the twisted product ("Melvin") case of $ds^2=-dt^2+d\rho^2+\rho^2(d\phi+qd\alpha)^2+R^2d\alpha^2$.

non-zero S^1 momentum:

$$t = \kappa\tau\,, \qquad \mathrm{Y} = Y_1 + iY_2 = \rho\, e^{i\phi}\,, \qquad \phi = \mathrm{w}\tau + k\sigma\,, \qquad \alpha = w\tau + m\sigma\ , \tag{3.1}$$

The free string equations are solved if the string radius is constant, ρ =const, and $\mathrm{w} = -k > 0$ (another solution has $\mathrm{w} = k$). The polar angle in $\mathbb{R}^2$ is thus $\phi = |k|(\tau - \sigma)$, i.e. $|k|$ may be interpreted as an integer "winding" in $\mathbb{R}^2$ (we shall assume that κ, m and w are positive and k is negative). m is an integer winding number in S^1 and the linear momentum along S^1 is also quantized. This configuration is thus a closed string wound along S^1 and having a left (or right) moving fluctuation along it (at fixed time the profile of the string is a spiral drawn on the (ϕ, α) torus). The corresponding string state should thus be 1/2 BPS in the superstring theory. Indeed, the conformal gauge constraints imply

$$\kappa^2 = 2\rho^2 k^2 + R^2(m^2 + w^2)\,, \qquad -\rho^2 k^2 + R^2 m w = 0\,, \quad \text{i.e.} \quad \kappa = R(m + w)\,, \tag{3.2}$$

and the string energy E, the spin S in $\mathbb{R}^2$ and the S^1 momentum J are given by

$$(E, S, J) = \frac{1}{\alpha'}(\mathcal{E}, \mathcal{S}, \mathcal{J})\,, \qquad \mathcal{E} = R(m + w)\,, \qquad \mathcal{S} = \rho^2|k| = R^2\frac{mw}{|k|}, \qquad \mathcal{J} = R^2 w\ , \tag{3.3}$$

where $\frac{1}{2\pi\alpha'}$ is the string tension. The constraints (3.2) expressed in terms of E,S,J,k are

$$\mathcal{E} = R^{-1}\mathcal{J} + R|k|\frac{\mathcal{S}}{\mathcal{J}}\,, \qquad k\mathcal{S} + m\mathcal{J} = 0\,. \tag{3.4}$$

If we now replace $\mathbb{R}^{1,2}$ with the AdS_3 subspace of AdS_5 and consider a similar string configuration in $AdS_3 \times S^1$ we obtain the circular solution [24] in $AdS_5 \times S^5$. It, however, will no longer be a BPS state, and will not reduce to the above flat-space solution in the limit of large radius of $AdS_5 \times S^5$. Indeed, here the radii of AdS_5 and S^5 are the same (with $\frac{R^2}{\alpha'} = \sqrt{\lambda}$), so that taking R large to approximate AdS_3 by $\mathbb{R}^{1,2}$ would also make the radius of S^1 large, which would correspond to sending the energy of the flat-space solution (3.4) to infinity.

Let us now review this circular solution of the $AdS_5 \times S^5$ equations following from (2.1) by using the conformal gauge. It is characterised by having constant Lagrange multipliers $\Lambda, \tilde\Lambda$ and constant induced metric. Setting $\mathrm{Y}_2 = 0$, $\mathrm{X}_2, \mathrm{X}_3 = 0$ (i.e. specialising to the SL(2) sector (2.12), (2.13) or to the motion in $AdS_3 \times S^1 \subset AdS_5 \times S^5$) one finds:[9]

$$\mathrm{Y}_0 = \mathrm{r}_0\, e^{i\kappa\tau}\,, \qquad \mathrm{Y}_1 = \mathrm{r}_1\, e^{i\mathrm{w}\tau + ik\sigma}\,, \qquad \mathrm{X}_1 = e^{iw\tau + im\sigma}\,, \tag{3.5}$$

$$\mathrm{r}_0 \equiv \cosh\rho_0\,, \qquad \mathrm{r}_1 \equiv \sinh\rho_0\,, \qquad \mathrm{r}_0^2 - \mathrm{r}_1^2 = 1\,, \tag{3.6}$$

where ρ_0 is a constant radius of the circular string, k and m are the winding numbers in the 2-plane in AdS_3 and in S^1, and w (which is no longer equal to k as in the flat space) and w are rotation frequencies of the string along itself.[10]

[9]The AdS_5 part of this solution can also be expressed in the Poincare coordinates ($ds^2 = \frac{dz^2 + dx_m dx_m}{z^2}$) as follows: $x_0 = \tan\kappa\tau$, $x_1 = \tanh\rho_0\, \frac{\cos(\mathrm{w}\tau + k\sigma)}{\cos\kappa\tau}$, $x_2 = \tanh\rho_0\, \frac{\sin(\mathrm{w}\tau + k\sigma)}{\cos\kappa\tau}$, $x_3 = 0$, $z = \frac{1}{\cosh\rho_0 \cos\kappa\tau}$.

[10]Note that this solution cannot be analytically continued directly to a physical solution in the SU(2) sector as was done for a folded string solution in [7] since that would introduce windings in the transformed time ($\alpha \to t$); one will need also to redefine the world-sheet coordinates to keep both AdS_5 time coordinates single-valued.

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Since the angles $\phi_1 = \mathrm{w}\tau + k\sigma$ and $\alpha = w\tau + m\sigma$ in (2.12),(2.13) (which correspond to isometries of the metric and thus enter the string lagrangian only through their derivatives) are linear in the world-sheet coordinates, their derivatives are constant, and thus the induced metric and also all the coefficients in the fluctuation lagrangian near this solution (to all orders in the fluctuations) are constant. The equations of motion imply:

$$\mathrm{w}^2 = \kappa^2 + k^2\,, \qquad w^2 = \nu^2 + m^2\,, \qquad \nu^2 = -\Lambda, \qquad \kappa^2 = \tilde{\Lambda}\,. \tag{3.7}$$

In terms of the non-zero charges

$$(E, S, J) = \sqrt{\lambda}(\mathcal{E}, \mathcal{S}, \mathcal{J})\,, \qquad \mathcal{E} = \mathrm{r}_0^2\kappa\,, \qquad \mathcal{S} = \mathrm{r}_1^2\mathrm{w}\,, \qquad \mathcal{J} = w\,, \tag{3.8}$$

the conformal gauge constraints are (we assume that $\mathcal{S}, \mathcal{J}, m > 0$ and $k < 0$)

$$2\kappa\mathcal{E} - \kappa^2 = 2\sqrt{k^2 + \kappa^2}\mathcal{S} + \mathcal{J}^2 + m^2\,, \tag{3.9}$$

$$k\mathcal{S} + m\mathcal{J} = 0\,, \tag{3.10}$$

and (3.6) gives also

$$\frac{\mathcal{E}}{\kappa} - \frac{\mathcal{S}}{\sqrt{k^2 + \kappa^2}} = 1\,. \tag{3.11}$$

As implied by these relations, there are only three independent parameters, e.g., $\mathcal{S}, \mathcal{J}, k$ (which are useful for comparison with gauge theory), or κ, r_1, k (which are useful for finding fluctuation frequencies discussed below in section 4). Eqs. (3.9), (3.10), (3.11) imply that in terms of κ, k, r_1

$$\nu^2 = \sqrt{\mathcal{J}^2 - m^2} = \sqrt{(\kappa^2 - 2k^2\mathrm{r}_1^2)^2 - 4k^2\mathrm{r}_1^4(k^2 + \kappa^2)}\,, \qquad m^2 = \frac{1}{2}\left(\kappa^2 - 2k^2\mathrm{r}_1^2 - \nu^2\right). \tag{3.12}$$

In terms of $\mathcal{J}, \mathcal{S}$ and k we find that for large $\mathcal{J}$ and $\mathcal{S}$ with fixed $u \equiv \frac{\mathcal{S}}{\mathcal{J}}$ and k [11] $\nu = \sqrt{\mathcal{J}^2 - k^2u^2} = \mathcal{J} - \frac{k^2}{2\mathcal{J}}u^2 + \cdots$, and

$$\kappa = \mathcal{J} + \frac{k^2}{2\mathcal{J}}u(2+u) - \frac{k^4}{8\mathcal{J}^3}u(4 + 12u + 8u^2 + u^3) + \cdots\,, \qquad u \equiv \frac{\mathcal{S}}{\mathcal{J}}\,, \tag{3.13}$$

$$\mathrm{w} = \sqrt{\kappa^2+k^2} = \mathcal{J} + \frac{k^2}{2\mathcal{J}}(1+u)^2 + \cdots\,, \qquad \mathrm{r}_1^2 = \frac{\mathcal{S}}{\sqrt{\kappa^2+k^2}} = u - \frac{k^2}{2\mathcal{J}^2}u(1+u)^2 + \cdots\,, \tag{3.14}$$

and the energy $E = \sqrt{\lambda}\,\mathcal{E}(\mathcal{S}, \mathcal{J}, k) = E(S, J, k; \lambda)$ becomes (cf. (1.1))

$$E = J(1+u)\left[1 + \frac{\lambda k^2}{2J^2}u - \frac{\lambda^2 k^4}{8J^4}u(1 + 3u + u^2) + O\left(\frac{\lambda^3}{J^6}\right)\right], \qquad u \equiv \frac{S}{J}\,. \tag{3.15}$$

Given that the string energy looks like a regular expansion in λ, it was suggested in [24] (by analogy with previous results for string solutions in SU(2) and SU(3) sectors [5, 46, 6, 8, 47]) and later verified in [1] that the leading order λ term in (3.15) can be reproduced as a one-loop anomalous dimension of a SYM operator from the SL(2) sector.

[11] In this case the string motion is "fast" [34] meaning that each point of the string moves at almost the speed of light and thus the induced metric degenerates.

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Finally, let us discuss how the above winding circular string appears (at leading order in $\mathcal{J}$) as a solution of the "Landau-Lifshitz" equation (2.27).[12] Concentrating first on the S^5 part of the configuration we note that the world-sheet metric h_{ab} has, to leading order, the following components $h_{00} = -1$, $h_{01} = -B_1$, $h_{11} = \mathcal{J}^2$, $h = \mathcal{J}^2 + B_1^2$. Then (2.20) and (2.23) imply (after the rescaling of τ so that $t = \mathcal{J}\tau$) that $\alpha = \mathcal{J}\tau + m\sigma$, which is the same as the phase of X_1 in (3.5). Turning to the AdS_5 part, the lagrangian corresponding to the leading-order term in (2.28) is (after rescaling back $\tau \to \frac{1}{\mathcal{J}^2}\tau$)

$$L = \dot{\eta} \sinh^2 \rho - \frac{1}{2\mathcal{J}^2} \left(\rho'^2 + \eta'^2 \sinh^2 \rho \ \cosh^2 \rho\right), \tag{3.16}$$

where we used (2.13) and introduced $\eta = \phi_1 - t$ (dot and prime are ∂_τ and ∂_σ).[13] The corresponding equations of motion have solution with $\rho = \rho_0$=const provided $\eta'' = 0$. The solution for η is then found to be $\eta = q\tau + k\sigma$, $q = \frac{k^2}{2\mathcal{J}^2}(1 + 2\mathrm{r}_1^2)$, implying that $\phi_1 = \mathcal{J}\tau + k\sigma$, which indeed is the leading order form of ϕ_1 in (3.5). Using (2.29) one can compute the space-time energy which reproduces the order λ term in (3.15). Also, the "winding" constraint (2.33) leads, to the leading order, to the same condition that follows in general from the conformal gauge constraint $m\mathcal{J} + k\mathcal{S} = 0$.

4. One-loop correction to energy of circular solution

As was already mentioned, for all "homogeneous" circular solutions [24] like the one of the previous section (3.5) the fluctuation lagrangian has constant coefficients (to all orders in fluctuation fields) [24, 11], a property shared with the BMN case where one expands near a point-like geodesic [4, 16, 17]. That means, in particular, that the spectrum of quadratic fluctuations can be found explicitly (and one can, in principle, also compute subleading corrections to their energies as in the BMN case in [17]). The general procedure of how to do that for a generic quadratic 2-d lagrangian of the type ($p = 1, \ldots, N$, and K, W, M are constant matrices)

$$L_2 = \dot{x}_p^2 - x_p'^2 + K_{pq} x_p \dot{x}_q + W_{pq} x_p x_q' + M_{pq} x_p x_q \,, \tag{4.1}$$

was discussed in [49, 5, 25].[14] Assuming that the fields are periodic in σ, one sets

$$x^p(\tau, \sigma) = \sum_{n=-\infty}^{\infty} x_n^p(\tau) e^{in\sigma} \,, \qquad x_n^p(\tau) = \sum_{I=1}^{2N} A_{I,n}^p e^{i\omega_{I,n}\tau} \,, \tag{4.2}$$

where I labels $2N$ "phase-space" directions. Then the field equations lead to a system of linear homogeneous equations for $A_{I,n}^p$, i.e. $F_{pq} A_{I,n}^q = 0$, where F_{pq} depends on $n, \omega_{I,n}$ and the coefficients in the lagrangian (4.1). It has solutions provided $\det F_{pq} = 0$, which gives

[12] Circular and folded string solutions of the Landau-Lifshitz equations in the (S, J) sector were also discussed in [48]. In the case of the circular solution, it is important not to do field redefinitions that may contradict the required single-valuedness condition for the AdS_5 time $t(\sigma + 2\pi, \tau) = t(\sigma, \tau)$.

[13] Note that after the field redefinition $\eta \to -2\eta$ this lagrangian is the same as [33, (2.31)].

[14] Such lagrangian is readily obtained for the bosonic fluctuations; similar first order or second order one is found for the fermionic fields.

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an order $2N$ polynomial equation for the characteristic frequencies. As follows from the reality conditions, which translate into $F^T(\omega_{I,n}, n) = F(-\omega_{I,n}, -n)$, the zero-mode ($n = 0$) frequencies come in pairs, i.e. $\omega_{I,0} = \{\pm\omega_{p,0}\}$ ($p = 1, \ldots, N$). The non-zero frequencies can be labelled so that they satisfy $\omega_{I,n} = -\omega_{I,-n}$, i.e. can also be paired (and associated to creation and annihilation operators upon quantization). Then the one-loop correction to the 2d ground state energy is given by the following sum of the non-trivial half of the characteristic frequencies [49],

$$E_{2\mathrm{d}} = \frac{1}{2}\sum_{p=1}^{N} \hat{\omega}_{p,0} + \frac{1}{2}\sum_{n=1}^{\infty}\sum_{I=1}^{2N} \hat{\omega}_{I,n}\,. \tag{4.3}$$

Here

$$\hat{\omega}_{p,0} = \mathrm{sign}(C_p^B)\omega_{p,0}\,, \qquad \hat{\omega}_{I,n} = \mathrm{sign}(C_{I,B}^{(n)})\omega_{I,n}\,, \tag{4.4}$$

$$C_p^B = \frac{1}{2m_{11}(\omega_{p,0})\omega_{p,0}\prod_{q\neq p}(\omega_{p,0}^2 - \omega_{q,0}^2)}, \quad C_{I,B}^{(n)} = \frac{1}{m_{11}(\omega_{I,n})\prod_{J\neq I}(\omega_{I,n} - \omega_{J,n})}\,, \tag{4.5}$$

where m_{11} is a minor of F, i.e. the determinant of the matrix obtained from F by removing the first row and first column.[15]

In the case of the circular string solution the number of transverse fluctuations is $N = 8$ and since we are to start with the superstring action [50] we also have the fermionic modes (see [4, 5, 25] for details).[16] Since in (3.5) one has $t = \kappa\tau$, the 1-loop correction to the space-time energy E_1 is then given by

$$E_1 = \frac{1}{\kappa}E_{2\mathrm{d}} = \frac{1}{2\kappa}\left[\sum_{p=1}^{8}\left(\hat{w}_{p,0}^B - \hat{\omega}_{p,0}^F\right) + \sum_{n=1}^{\infty}\sum_{I=1}^{16}\left(\hat{\omega}_{I,n}^B - \hat{\omega}_{I,n}^F\right)\right]. \tag{4.6}$$

Finally, one would like to expand the parameters in E_1 at large $\mathcal{J}$ and compare the leading asymptotics of E_1 with the subleading term in the SYM anomalous dimension (1.2).

The first step is thus to find, as in [25], the bosonic and fermionic lagrangians for small fluctuations near the circular solution (3.5) and then to compute the corresponding characteristic frequencies. This will be done in the next two subsections 4.1 and 4.2, where we shall also verify the stability of the solution, i.e. the reality of the characteristic frequencies and thus of E_1 for large enough angular momenta. In section 4.3 we shall check the UV finiteness of (4.6) which is a consequence of the conformal invariance of the $AdS_5 \times S^5$ superstring theory [50, 5, 25]. We shall also note a possibility of evaluating

[15] For example, for a single real ($N = 1$) 2d field with mass μ the characteristic equation is $\omega^2 = n^2 + \mu^2$, so that $E_{2\mathrm{d}} = \frac{1}{2}\sum_{n=-\infty}^{\infty}\sqrt{n^2+\mu^2} = \frac{1}{2}\mu + \sum_{n=1}^{\infty}\sqrt{n^2+\mu^2}$. To see that this is the same as (4.3) we note that in the case of a free massive field $\mathrm{sign}(C_p^B)\omega_{p,0} = |\omega_{p,0}|$, $\mathrm{sign}(C_{I,B}^{(n)})\omega_{I,n} = |\omega_{I,n}|$. (Note that this relation is not true in general — it is true only when half of the frequencies are positive and half are negative for $n > 0$.) This can be seen by writing the matrix F for a system of two first-order equations in time: $F_{11} = n^2 + \mu^2$, $F_{12} = i\omega_{I,n}$, $F_{21} = -i\omega_{I,n}$, $F_{22} = 1$. Then (4.3) gives again $\frac{1}{2}\mu + \sum_{n=1}^{\infty}\sqrt{n^2+\mu^2}$ since for a single massive field with mass μ: $|\omega_{1,n}| = |\omega_{2,n}| = \sqrt{n^2+\mu^2}$.

[16] As we shall see below, here both the bosonic and the fermionic 2d fields are periodic in σ so that the sum over n is over the integers.

the asymptotics of E_1 by interchanging summation over modes with large spin expansion. Since the analytic evaluation of the sums involved does not appear to be possible we shall then follow ref. [27] and compute the leading asymptotics of E_1 at large J and fixed S/J using a numerical method.

4.1 Bosonic frequencies

Below we shall review the derivation of the bosonic characteristic frequencies for the solution (3.5) in the conformal gauge following [24]. Let us consider the S^5-directions first. In general, starting with (2.2)–(2.5) and expanding $\mathrm{X}_i \to \mathrm{X}_i + \tilde{\mathrm{X}}_i$ one finds the quadratic fluctuation lagrangian in terms of the 3 complex fields

$$\tilde{L}_S = -\frac{1}{2}\,\partial_a \tilde{\mathrm{X}}_i \partial^a \tilde{\mathrm{X}}_i^* + \frac{1}{2}\Lambda\,\tilde{\mathrm{X}}_i \tilde{\mathrm{X}}_i^*\,, \qquad \sum_{i=1}^{3}(\mathrm{X}_i \tilde{\mathrm{X}}_i^* + \mathrm{X}_i^* \tilde{\mathrm{X}}_i) = 0\,. \tag{4.7}$$

For the solution (3.5) $\tilde{\mathrm{X}}_2, \tilde{\mathrm{X}}_3$ are decoupled and represented by 4 real massive (with mass ν, see (3.7)) 2d fields, i.e. they contribute to the energy (4.6) the term

$$\frac{1}{2\kappa}\left(4\nu + 2\sum_{n=1}^{\infty} 4\sqrt{n^2+\nu^2}\right), \qquad \nu^2 = \mathcal{J}^2 - m^2\,. \tag{4.8}$$

The lagrangian for the fluctuations in the AdS_5 directions is

$$\tilde{L}_{AdS} = -\frac{1}{2}\,\partial_a \tilde{\mathrm{Y}}^r \partial^a \tilde{\mathrm{Y}}_r^* - \frac{1}{2}\tilde{\Lambda}\,\tilde{\mathrm{Y}}^r \tilde{\mathrm{Y}}_r^*\,, \qquad \sum_{r=0}^{2}(\mathrm{Y}^r \tilde{\mathrm{Y}}_r^* + \mathrm{Y}_r^* \tilde{\mathrm{Y}}^r) = 0\,. \tag{4.9}$$

For the solution (3.5) the mode $\tilde{\mathrm{Y}}_2$ is a decoupled massive complex field or two real fields, i.e. it contributes to (4.6) as

$$\frac{1}{2\kappa}(2\kappa + 2\sum_{n=1}^{\infty} 2\sqrt{n^2+\kappa^2}\ )\,. \tag{4.10}$$

Setting for the remaining fluctuation fields

$$\tilde{\mathrm{X}}_1 = e^{iw\tau + im\sigma}(g_1 + if_1)\,, \qquad \tilde{\mathrm{Y}}_0 = e^{i\kappa\tau}(G_0 + iF_0)\,, \qquad \tilde{\mathrm{Y}}_1 = e^{i\mathrm{w}\tau + ik\sigma}(G_1 + iF_1)\,, \tag{4.11}$$

we can solve the conditions in (4.7) and (4.9) as

$$g_1 = 0\,, \qquad\qquad G_0 = \frac{\mathrm{r}_1}{\mathrm{r}_0}G_1\,, \tag{4.12}$$

so that the lagrangian for the 4 real fields (f_1, F_0, F_1, G_1) becomes

$$\begin{aligned}\tilde{L} &= \frac{1}{2}(\dot{f}_1^2 - f_1'^2) - \frac{1}{2}(\dot{F}_0^2 - F_0'^2) + 2\frac{\mathrm{r}_1}{\mathrm{r}_0}\kappa F_0 \dot{G}_1 + \\ &+ \frac{1}{2}\left(\dot{F}_1^2 + \frac{1}{\mathrm{r}_0^2}\dot{G}_1^2 - F_1'^2 - \frac{1}{\mathrm{r}_0^2}G_1'^2\right) - 2\mathrm{w}F_1\dot{G}_1 + 2kF_1G_1'\,.\end{aligned} \tag{4.13}$$

Note that f_1 couples to (F_0, F_1, G_1) through the conformal gauge conditions.[17] After an appropriate linear field redefinitions two of these four modes become massless and decouple, and their contribution to the 1-loop effective action (which does not depend on the parameters of the background) gets cancelled by the two conformal ghost contributions. The upshot, as in [25], is that one can simply ignore the constraints on fluctuations implied by the conformal gauge conditions and omit two massless longitudinal modes.[18] Then the 4 characteristic frequencies corresponding to the two remaining real 2d fields can be found from the following matrix F

$$\begin{pmatrix} \frac{\omega^2-n^2}{1+\mathrm{r}_1^2} + \frac{4k^2\kappa^2\mathrm{r}_1^2}{A} & -\frac{2ik\mathrm{r}_1}{A}[\omega(k\mathrm{w}-mw)+n(km-w\mathrm{w})] \\ \frac{2ik\mathrm{r}_1}{A}[\omega(k\mathrm{w}-mw)+n(km-w\mathrm{w})] & (\omega^2-n^2)\left(1-\frac{\mathrm{w}^2-m^2}{A}\right) \end{pmatrix} \tag{4.14}$$

where $A = \frac{1}{2}(\sqrt{\kappa^4 - 4\kappa^2k^2\mathrm{r}_1^2(1+\mathrm{r}_1^2)} - \kappa^2)$. The $\det F = 0$ condition gives the quartic equation (we use κ, r_1, k as independent parameters) [24]

$$(\omega^2 - n^2)^2 + 4\mathrm{r}_1^2\kappa^2\omega^2 - 4(1+\mathrm{r}_1^2)(\sqrt{\kappa^2+k^2}\,\omega - kn)^2 = 0\,. \tag{4.15}$$

The 4 roots $\omega = \{\omega_{I,n}\}$ $(I = 1,2,3,4)$ are the characteristic frequencies in the discussion at the beginning of this section (here $\sum_{I=1}^4 \omega_{I,n} = 0$ since there is no ω^3 term in (4.15)). Note that since (4.15) is invariant under $\omega \to -\omega,\ n \to -n$, we have, as required, the pairing of $n \neq 0$ frequencies: $\omega_{I,n} = -\omega_{I,-n}$. The zero modes are given by (using (3.7))

$$\omega_{I,0} = \{0, 0, \omega_0, -\omega_0\}\,, \qquad \omega_0 = 2\sqrt{\kappa^2 + (1+\mathrm{r}_1^2)k^2}\,, \tag{4.16}$$

so that the contribution of these "mixed" modes to the bosonic part of (4.6) is thus

$$\frac{1}{2\kappa}\left(\omega_0 + \sum_{n=1}^{\infty}\sum_{I=1}^{4} \mathrm{sign}(C_{I,B}^{(n)})\omega_{I,n}\right). \tag{4.17}$$

The relevant part of the minor m_{11} for computing the signs of $C_{I,B}^{(n)}, C_p^B$ is $m_{11} \sim (\omega^2 - n^2)$, since $\left(1 - \frac{\mathrm{w}^2-m^2}{A}\right)$ is always positive. While the 4 roots $\omega_{I,n}$ of the quartic equation (4.15) can be written down explicitly, the resulting sum in (4.25) cannot be done analytically for generic values of the parameters. Below we shall discuss a way to evaluate the sum in the large spin limit we are interested in.

In order for the circular string configuration to be stable all bosonic frequencies must be real, and this is not a priori obvious for the solutions of (4.15). If one expands $\omega_{I,n}$ at large κ or large $\mathcal{J}$ (with n and $\mathrm{r}_1^2 \approx u, k$ being fixed, cf. (3.13)), one finds that the

[17] The conformal gauge constraints have the form $2\mathrm{r}_1k^2G_1 - \mathrm{r}_0\kappa\dot{F}_0 + \mathrm{r}_1(\mathrm{w}\dot{F}_1 + kF_1') + w\dot{f}_1 + mf_1' = 0$, $-\mathrm{r}_0\kappa F_0' + 2\mathrm{r}_1\mathrm{w}kG_1 + \mathrm{r}_1(\mathrm{w}F_1' + k\dot{F}_1) + wf_1' + m\dot{f}_1 = 0$.

[18] The same conclusion is found by starting with the Nambu action and using static gauge condition on the fluctuation fields.

frequencies are real [24][19]

$$\omega_{I=1,2;n} = \frac{n}{2\mathcal{J}}\left[2k(1+\mathrm{r}_1^2) \pm \sqrt{n^2+4k^2\mathrm{r}_1^2(1+\mathrm{r}_1^2)}\right] + O\left(\frac{1}{\mathcal{J}^3}\right), \qquad (4.18)$$

$$\omega_{I=3,4;n} = \pm 2\mathcal{J} \pm \frac{1}{2\mathcal{J}}[n^2 \mp 2kn(1+\mathrm{r}_1^2) + 2k^2(1+3\mathrm{r}_1^2+\mathrm{r}_1^4)] + O\left(\frac{1}{\mathcal{J}^3}\right). \qquad (4.19)$$

As a result, the expression for E_1 is real and its comparison to gauge-theory anomalous dimension is unambiguous (in contrast to the case of the circular solution from SU(2) sector discussed in [25, 27] which is unstable for any $J_1 = J_2$). Using the above expressions of the frequencies at large $\mathcal{J}$ the corresponding signs of $C^{(n)}_{I,B}$ are found to be: $\mathrm{sign}(C^{(n)}_{1,B}) = \mathrm{sign}(C^{(n)}_{3,B}) = +1$ and $\mathrm{sign}(C^{(n)}_{2,B}) = \mathrm{sign}(C^{(n)}_{4,B}) = -1$. They are needed later to compute the energy E_1 in the same limit.

Let us mention that out of all the fluctuation frequencies only the two in (4.18) scale as $\frac{a}{\mathcal{J}}$ at large $\mathcal{J}$, while all other (including the fermionic frequencies discussed below) scale as $\mathcal{J} + \frac{b}{\mathcal{J}}$. The two bosonic frequencies of the former type have been reproduced from the Landau-Lifshitz action in (C.1): they correspond to deformations along the "intrinsic" SL(2) sector directions ρ and ϕ_1 in (2.13). These two are then the ones that should be "seen" also on the SL(2) spin chain side [33, 1] (to reproduce other frequencies one is to embed the solution into a larger sigma model or spin chain sector).

4.2 Fermionic frequencies

The quadratic part of the $AdS_5 \times S^5$ superstring lagrangian evaluated on a bosonic solution has a simple form (see [50, 4, 5, 25] for details)

$$L_F = i\left(\eta^{ab}\delta^{IJ} - \epsilon^{ab}s^{IJ}\right)\bar\theta^I\rho_a D_b\,\theta^J, \qquad \rho_a \equiv \Gamma_A e^A_a, \qquad e^A_a \equiv E^A_\mu(\mathcal{X})\partial_a\mathcal{X}^\mu, \qquad (4.20)$$

where $I, J = 1, 2$, $s^{IJ} = \mathrm{diag}(1, -1)$, ρ_a are projections of the ten-dimensional Dirac matrices and $\mathcal{X}^\mu$ are the coordinates of the AdS_5 space for $\mu = 0, 1, 2, 3, 4$ and the coordinates of S^5 for $\mu = 5, 6, 7, 8, 9$. The covariant derivative is given by

$$D_a\theta^I = \left(\delta^{IJ}\mathrm{D}_a - \frac{i}{2}\epsilon^{IJ}\Gamma_*\rho_a\right)\theta^J, \qquad \Gamma_* \equiv i\Gamma_{01234}, \quad \Gamma_*^2 = 1, \qquad (4.21)$$

where $\mathrm{D}_a = \partial_a + \frac{1}{4}\omega_a^{AB}\Gamma_{AB}$, $\omega_a^{AB} \equiv \partial_a\mathcal{X}^\mu\omega_\mu^{AB}$. Fixing the κ-symmetry by the same condition as in [25] $\theta^1 = \theta^2 = \theta$ one gets

$$L_F = -2i\bar\theta D_F\theta, \qquad D_F = -\rho^a\mathrm{D}_a - \frac{i}{2}\epsilon^{ab}\rho_a\Gamma_*\rho_b. \qquad (4.22)$$

Labelling the coordinates as follows (cf. (2.6), (2.7)):

$$\begin{array}{ccccccccccc} \mu: & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \mathcal{X}^\mu: & t & \rho & \theta & \phi_1 & \phi_2 & \gamma & \varphi_1 & \psi & \varphi_2 & \varphi_3 \end{array} \qquad (4.23)$$

[19] They are also always real for large n. Numerical analysis shows that for any given n there always exists a large enough $\mathcal{J}$ for which all 4 frequencies are real. For small values of $\mathcal{J} \sim 1$ and some small n the solutions of (4.15) become complex. The stability of the circular (S, J) solution (3.5) at large spin $\mathcal{J}$ is similar to the stability of the 3-spin S^5 solution $(J_1 = J_2, J_3)$ which is also stable for large enough J_3 [25].

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one finds (using (3.5),(3.6)) that the non-trivial components of the Lorentz connection ω_a^{AB} are $\omega_\tau^{10} = -\kappa \mathrm{r}_1$, $\omega_\tau^{13} = -\mathrm{w}\mathrm{r}_0$, $\omega_\sigma^{13} = -k\mathrm{r}_0$, and then

$$\begin{aligned} D_F &= (\kappa \mathrm{r}_0\, \Gamma_0 + \mathrm{w}\mathrm{r}_1 \Gamma_3 + w\, \Gamma_6) \left(\partial_\tau - \frac{1}{2}\kappa \mathrm{r}_1 \Gamma_{10} - \frac{1}{2}\mathrm{w}\mathrm{r}_0\, \Gamma_{13}\right) - \\ &\quad - (k\mathrm{r}_1\, \Gamma_3 + m\, \Gamma_6) \left(\partial_\sigma - \frac{1}{2}k\mathrm{r}_0 \Gamma_{13}\right) + k\kappa \mathrm{r}_1 \mathrm{r}_0 \Gamma_{124}\,. \end{aligned} \tag{4.24}$$

We observe that in contrast to the case of the S^5 circular string solution considered in [25, 27] here D_F does not depend on σ, and thus one does not need to apply a local rotation to eliminate this σ dependence (which in [25, 27] was making the rotated fermions antiperiodic in σ). To simplify D_F it is useful to do constant rotations in (36)-plane and (06)-plane (see Appendix B) which do not affect the periodicity of the fermions, so the sum over the fermionic characteristic frequencies will go over integer n as in (4.6).

Since θ is a Majorana-Weyl 10d spinor, we end up with the total of 16 characteristic frequencies (i.e. $I = 1, \ldots, 16$) corresponding to 8 physical 2d fermionic modes. Their contribution to (4.6) expressed in terms of the three independent parameters $\kappa, \mathrm{r}_1^2, k$ is found to be (see appendix B for details)

$$-\frac{1}{2\kappa}\left(8\tilde{\omega}_0 + 8\sum_{n=1}^{\infty}\left[\sqrt{(n+c)^2 + a^2} + \sqrt{(n-c)^2 + a^2}\,\right]\right), \tag{4.25}$$

where

$$\begin{aligned} \tilde{\omega}_0 &= \sqrt{c^2 + a^2}\,, && (4.26)\\ a^2 &= \frac{1}{2}(\kappa^2 + \nu^2)\,, \qquad \nu^2 = \sqrt{(\kappa^2 - 2k^2 \mathrm{r}_1^2)^2 - 4k^2 \mathrm{r}_1^4 (\kappa^2 + k^2)}\,, && (4.27)\\ c &= \frac{1}{2}\kappa\left[1 + \frac{2k^2(1 + \mathrm{r}_1^2)}{\kappa^2 - \nu^2}\right]\sqrt{\frac{\kappa^2 - \nu^2 - 2k^2 r_1^2}{2(\kappa^2 + k^2)}}\,. && (4.28) \end{aligned}$$

The large $\mathcal{J}$ expansion of the fermionic frequencies in (4.25) is (the expansion of $\tilde{\omega}_0$ is obtained by setting $n = 0$)

$$\sqrt{(n+c)^2 + a^2} + \sqrt{(n-c)^2 + a^2} = 2\mathcal{J} + \frac{4n^2 + k^2(1 + 6u + u^2)}{4\mathcal{J}} + O\left(\frac{1}{\mathcal{J}^3}\right). \tag{4.29}$$

4.3 Evaluation of E_1

The final expression for E_1 in (4.6) is thus given by the sum of (4.8), (4.10), (4.17) and (4.25). Let us split E_1 into the contribution of zero ($n = 0$) and non-zero ($n > 1$) modes

$$E_1 = E_1^{(0)} + \bar{E}_1\,, \qquad E_1^{(0)} = \frac{1}{2\kappa}(4\nu + 2\kappa + \omega_0 - 8\tilde{\omega}_0)\,, \tag{4.30}$$

$$\begin{aligned} \bar{E}_1 &= \frac{1}{\kappa}\sum_{n=1}^{\infty}\Bigg(4\sqrt{n^2 + \nu^2} + 2\sqrt{n^2 + \kappa^2} + \frac{1}{2}\sum_{I=1}^{4}\mathrm{sign}(C_{I,B}^{(n)})\omega_{I,n} - \\ &\quad - 4\left[\sqrt{(n+c)^2 + a^2} + \sqrt{(n-c)^2 + a^2}\,\right]\Bigg)\,, \end{aligned} \tag{4.31}$$

It is easy to check that the sum over n is convergent. Using the large n asymptotics of

$$\sqrt{n^2+\nu^2} = |n| + \frac{\nu^2}{2|n|} + \cdots\,, \qquad \sqrt{n^2+\kappa^2} = |n| + \frac{\kappa^2}{2|n|} + \cdots\,, \tag{4.32}$$

and of the 4 solutions of (4.15)

$$\omega_{I=1,3;n} = |n|\left(1 \pm \frac{1}{|n|}\sqrt{(1+\mathrm{r}_1^2)(\sqrt{\kappa^2+k^2}-k)^2 - \mathrm{r}_1^2\kappa^2} + \frac{\kappa^2}{2n^2} + \cdots\right), \tag{4.33}$$

$$\omega_{I=2,4;n} = -|n|\left(1 \mp \frac{1}{|n|}\sqrt{(1+\mathrm{r}_1^2)(\sqrt{\kappa^2+k^2}+k)^2 - \mathrm{r}_1^2\kappa^2} + \frac{\kappa^2}{2n^2} + \cdots\right), \tag{4.34}$$

we find

$$\bar{E}_1 = \frac{1}{\kappa}\sum_{n=1}^{\infty}\left(\left[8n + \frac{2(\kappa^2+\nu^2)}{n} + \cdots\right] - \left[8n + \frac{4a^2}{n} + \cdots\right]\right). \tag{4.35}$$

We used here the signs of $C^{(n)}_{I,B}$ for the above expressions of frequencies $\omega_{I=1,2,3,4;n}$, which are $\mathrm{sign}(C^{(n)}_{1,B}) = \mathrm{sign}(C^{(n)}_{3,B}) = +1$, $\mathrm{sign}(C^{(n)}_{2,B}) = \mathrm{sign}(C^{(n)}_{4,B}) = -1$.[20] Thus the divergent terms in the sum over n indeed cancel between the bosonic and fermionic contributions: according to (4.27) $a^2 = \frac{1}{2}(\kappa^2+\nu^2)$.

We are interested in the large $\mathcal{J}$ expansion of $E_1(\mathcal{J},\mathcal{S},k)$ at fixed winding number k (or m) and fixed $\mathcal{S}/\mathcal{J}$. Let us look first at the zero-mode part $E_1^{(0)}$ (4.30) which is known explicitly. Expanding it at large $\mathcal{J}$ we find

$$E_1^{(0)} = -\frac{k^2u(1+u)}{2\mathcal{J}^2} + O\left(\frac{1}{\mathcal{J}^4}\right), \qquad u = \frac{\mathcal{S}}{\mathcal{J}}\,. \tag{4.36}$$

Surprisingly, this already coincides with the result of [1] for the subleading $1/J$ correction in the 1-loop anomalous dimension (1.2) of the corresponding SYM operator.[21] In general, one expects that $\bar{E}_1$ in (4.31) has the following expansion

$$\bar{E}_1 = \frac{f_2\,(u,k)}{\mathcal{J}^2} + \frac{f_4(u,k)}{\mathcal{J}^4} + \cdots\,. \tag{4.37}$$

To find f_2 we are supposed to first do the sum in (4.31) and then expand the result at large $\mathcal{J}$. However, given that computing the sum explicitly for any $\mathcal{J}$ does not seem possible, one may attempt to first expand the frequencies at large $\mathcal{J}$ and then do the sum over n, separately for each $1/\mathcal{J}^k$ coefficient. This may not be consistent in general: the procedures of expanding in $1/\mathcal{J}$ and summing over n may not commute.[22]

[20] Not that exactly the same signs of $C^{(n)}_{I,B}$ we found before for the frequencies at large $\mathcal{J}$. But this is not necessarily true for any $\mathcal{J}$ and n.

[21] Similar observation applies to the SU(2) case with $J_1 = J_2$ considered in [27]: there the coefficient of the $1/\mathcal{J}^2$ term in the zero-mode part of the sum for the 1-loop energy also has the same form as the 1-loop gauge theory expression found in [26].

[22] Indeed, in taking $\mathcal{J}$ large in the frequencies one assumes that n is fixed, but n can take arbitrarily large values in the sum. The consistency of this procedure depends on how rapidly the expansion and the sum are converging: it is likely that the sums over n in the coefficients of higher $1/\mathcal{J}^k$ terms become divergent at large n, despite the fact that, as we have seen above, the sum in $\bar{E}_1$ is finite at fixed $\mathcal{J}$. Then in general one may not be able to trust the coefficient of the $1/\mathcal{J}^2$ term obtained by "first expanding, then doing the sum" procedure: one will need to resum the (divergent) expansion in $1/\mathcal{J}$.

u	0.5	0.6	0.7	0.8	0.9	1	1.1	1.2
$\tilde{f}_2(u,1)$	-0.56	-0.85	-1.22	-1.68	-2.23	-2.88	-3.65	-4.53

Table 1: Values of $\tilde{f}_2$ for $k = 1$.

Applying this procedure of first expanding the frequencies at large $1/\mathcal{J}$ one finds the expected $1/\mathcal{J}^2$ asymptotics, and moreover, that, remarkably, the coefficient of the $1/\mathcal{J}^2$ term is given by a *convergent* sum over n. Explicitly, expanding the bosonic and fermionic terms in (4.31) at large $\mathcal{J}$ we get

$$\begin{aligned}\bar{E}_1 &= \sum_{n=1}^{\infty}\left(8 + \frac{1}{2\mathcal{J}^2}[7n^2 + 2k^2(1+u)(1-4u) + n\sqrt{n^2+4k^2u(1+u)}] + O\left(\frac{1}{\mathcal{J}^4}\right)\right) - \\ &\quad - \sum_{n=1}^{\infty}\left(8 + \frac{1}{\mathcal{J}^2}[4n^2 + k^2(1+u)(1-3u)] + O\left(\frac{1}{\mathcal{J}^4}\right)\right) \\ &= \frac{\tilde{f}_2(u,k)}{\mathcal{J}^2} + O\left(\frac{1}{\mathcal{J}^4}\right). \end{aligned} \tag{4.38}$$

Here $\tilde{f}_2$ is given by the convergent series

$$\tilde{f}_2(u,k) = -\frac{1}{2}\sum_{n=1}^{\infty}\left[n^2 + 2k^2u(1+u) - n\sqrt{n^2+4k^2u(1+u)}\ \right], \tag{4.39}$$

and tilde indicates that this is a "naive" value of f_2. Note that $\tilde{f}_2$ is a continuous negative function of $x = 2k^2u(1+u)$ which vanishes at $x = 0$. Numerical evaluation of the sum in (4.39) with $n \leq 10^5$ gives the values in table 1 (for $k = 1$)

The coefficients of higher $O(1/\mathcal{J}^{2k})$ terms in the "naive" expansion (4.38) are found to be divergent, so the $1/\mathcal{J}$ expansion needs to be resummed, and the correct value of f_2 need not a priori coincide with $\tilde{f}_2$. Surprisingly, the results of the direct numerical evaluation of f_2 discussed below turn out to be essentially the same as the above results for $\tilde{f}_2$. That means that resummation of the higher $1/\mathcal{J}^4, \ldots$ terms in (4.38) does not change the coefficient of the $1/\mathcal{J}^2$ term: the "naive" coefficient $\tilde{f}_2$ is actually the correct one.

To compute the sum over n in (4.31) numerically at fixed $k, u = \frac{\mathcal{S}}{\mathcal{J}}$ and large $\mathcal{J}$ we have first simplified the parameters in the frequencies (but not expanding full $\omega_{I,n}$ at fixed n) in (4.8), (4.10), (4.15) and (4.25) using that $m = -ku$, $\nu^2 = \mathcal{J}^2 - k^2u^2$ and that κ has expansion given in (3.13). We then estimated the coefficient f_2 in (4.37) for various values of u (for simplicity we set $k = 1$ and considered u of order 1) by computing the sum over n with the upper limit $N = 5000$ and varying $\mathcal{J}$ in the interval $50 \leq \mathcal{J} \leq 1000$.[23] We confirmed the form of the expansion (4.37) with f_2 being *non-zero*. Its numerical values turned out to be the *same* as given in the table above for $\tilde{f}_2$. This demonstrates that for the given solution (and likely in some other similar cases where $\tilde{f}_2$ is given by a finite sum) the above procedure of first expanding in large $\mathcal{J}$ and then doing the sum is correct at order $1/\mathcal{J}^2$.[24]

[23] As was already mentioned, the solution is stable for large enough $\mathcal{J}$ (e.g., $\mathcal{J} > 50$) so the frequencies here are real.

[24] It would be desirable of course to find an analytic proof of correctness of this procedure.

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We conclude that the expression for the 1-loop correction to the string energy E_1, as obtained from (4.36),(4.37),(4.39), is given by

$$
\begin{aligned}
E_1 = &-\frac{k^2u(1+u)}{2\mathcal{J}^2} - \\
&-\frac{1}{2\mathcal{J}^2}\sum_{n=1}^{\infty}\left[n^2+2k^2u(1+u)-n\sqrt{n^2+4k^2u(1+u)}\,\right]+O(\frac{1}{\mathcal{J}^4})\,. \qquad (4.40)
\end{aligned}
$$

Acknowledgments

We are grateful to N. Beisert, S. Frolov, V. Kazakov, M. Kruczenski, O. Lunin, A. Mikhailov, B. Stefanski, J. Russo and K. Zarembo for useful discussions and comments. This work was supported by the DOE grant DE-FG02-91ER40690. The work of A.A.T. was also supported by the INTAS contract 03-51-6346 and RS Wolfson award.

A. Large $\mathcal{J}$ limit in 5-spin case

Here we shall consider the expansion of the string action in the case when the string may carry all 5 spins $(S_1, S_2, J_1, J_2, J_3)$, thus generalizing the discussion in [33] and in section 2. We shall still assume that isolating of the "fast" variables is done in terms of the 3+3 complex string coordinates in (2.4), (2.5), i.e. we shall define y and α as in (2.8). This will not cover the cases of more general string motions (like pulsations [51, 8]) discussed in [34, 35].

As follows from (2.9), the string lagrangian takes the form

$$L = -\frac{1}{2}\sqrt{-g}g^{ab}\left(-D_a y D_b y + D_a\alpha D_b\alpha + D_a V_r^* D_b V^r + D_a U_i^* D_b U^i\right). \qquad (A.1)$$

As in the (S,J)-sector (2.14), we shall again fix the U(1) symmetry by choosing y and V_r so that $y = t$ is the time coordinate (that amounts to shifting angles in (2.6) by $-t$). Two relevant combinations of charges are (cf. (2.17),(2.19))

$$\mathcal{E} - \mathcal{S}_1 - \mathcal{S}_2 = \int_0^{2\pi}\frac{d\sigma}{2\pi}\,p_y\,, \qquad \mathcal{J} \equiv \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3 = \int_0^{2\pi}\frac{d\sigma}{2\pi}\,p_\alpha\,, \qquad (A.2)$$

where $p_\alpha = -\sqrt{-g}g^{0a}D_a\alpha$, $p_y = p_t = -\sqrt{-g}g^{0a}D_a y$. Following [35], we shall introduce as in (2.20) the dual coordinate $\tilde{\alpha}$

$$\sqrt{-g}g^{ab}D_b\alpha = -\epsilon^{ab}\partial_b\tilde{\alpha}\,, \qquad (A.3)$$

so that after replacing α by $\tilde{\alpha}$ and eliminating the 2d metric, we get as in (2.21) ($C_a = -iU_i^*\partial_a U_i$, see (2.11))

$$
\begin{aligned}
L &= \epsilon^{ab}C_a\partial_b\tilde{\alpha} - \sqrt{h}\,, \qquad (A.4)\\
h_{ab} &= -D_a t D_b t + \partial_a\tilde{\alpha}\partial_b\tilde{\alpha} + D^*_{(a}V_r^* D_{b)}V^r + D^*_{(a}U_i^* D_{b)}U^i\,. \qquad (A.5)
\end{aligned}
$$

Choosing the same gauge as in (2.23), i.e. the static gauge in t and $\tilde{\alpha}$,

$$t = \tau\,, \qquad \tilde{\alpha} = \mathcal{J}\sigma\,, \tag{A.6}$$

we find $L = C_0 - \sqrt{h}$ with:

$$\begin{aligned} h = \Big| &(-B_1^2 + \mathcal{J}^2 + |D_1V_r|^2 + |D_1U_i|^2)(|D_0V_r|^2 + |D_0U_i|^2 - (1+B_0)^2) - \\ &- \left[-B_1(1+B_0) + \frac{1}{2}(D_0^*V_r^*D_1V^r + D_0^*U_i^*D_1U^i + \text{c.c.})\right]^2 \Big| \end{aligned} \tag{A.7}$$

Expanding at large $\mathcal{J}$ as in section 2 we get the action (2.24) with (2.26) replaced by

$$\begin{aligned} \tilde{L} = &-\mathcal{J}^2 - iU_i^*\partial_0U_i - iV_r^*\partial_0V^r - \frac{1}{2}|D_1V_r|^2 - \frac{1}{2}|D_1U_i|^2 + \\ &+ \frac{1}{8\mathcal{J}^2}\Big[(|D_1V_r|^2 + |D_1U_i|^2)^2 + 4(|D_0V_r|^2 + |D_0U_i|^2) - 4iV_r^*\partial_0V^r(|D_1V_r|^2 + |D_1U_i|^2) + \\ &+ 4iV_r^*\partial_1V^r(D_0^*V_r^*D_1V^r + D_0^*U_i^*D_1U^i + \text{c.c.})\Big] + O\left(\frac{1}{\mathcal{J}^4}\right). \end{aligned} \tag{A.8}$$

Finally, we can eliminate the time derivatives from the $1/\mathcal{J}^2$ term using leading-order equations and then obtain the 2d energy. Matching it with the charge $E - S_1 - S_2$ we obtain the space-time energy.

As in the (S, J)-sector, any possible winding of a closed string along the two angles of AdS_5 is assumed to be absorbed into $V_{1,2}$ (t must be single-valued), but in contrast to the case of the (S, J)-sector, now it is natural to absorb any possible windings in the three S^5 angles into U_i (see also [35]). Then we get (cf. (2.31)) $\int_0^{2\pi} \frac{d\sigma}{2\pi}\partial_1\alpha = 0$, and using (A.3) we obtain an additional constraint on the solutions

$$\int_0^{2\pi} \frac{d\sigma}{2\pi} C_1 = -\mathcal{J}\int_0^{2\pi}\frac{d\sigma}{2\pi}\frac{h_{01}}{\sqrt{h}}\,, \tag{A.9}$$

where

$$h_{01} = -B_1(1+B_0) + \frac{1}{2}(D_0^*V_r^*D_1V^r + D_0^*U_i^*D_1U^i + c.c.) \tag{A.10}$$

Rescaling $\tau \to \mathcal{J}^2\tau\,,\ \partial_0 \to \frac{1}{\mathcal{J}^2}\partial_0$ and expanding in large $\mathcal{J}$ this becomes (cf.(2.33))

$$\begin{aligned} -\int_0^{2\pi}\frac{d\sigma}{2\pi}\, iU_i^*\partial_1U_i = &\int_0^{2\pi}\frac{d\sigma}{2\pi}\, iV_r^*\partial_1V^r - \\ &- \frac{1}{2\mathcal{J}^2}\int_0^{2\pi}\frac{d\sigma}{2\pi}\Big[iV_r^*\partial_1V^r(|D_1V^r|^2 + |D_1U_i|^2) + \\ &+ (D_0^*V_r^*D_1V^r + D_0^*U_i^*D_1U^i + \text{c.c.})\Big] + O\left(\frac{1}{\mathcal{J}^4}\right). \end{aligned} \tag{A.11}$$

One can again eliminate the time derivatives here using the equations of motion.

B. More on fermionic frequencies

The fermionic operator given in (4.24) can be simplified (as in [25]) by performing appropriate rotations in the (36) and (06)-planes:

$$S_{36}S_{06} = e^{-\frac{1}{2}p\Gamma_3\Gamma_6}e^{-\frac{1}{2}q\Gamma_0\Gamma_6}\,, \qquad \sin p = \frac{m}{\sqrt{m^2+k^2\mathrm{r}_1^2}}\,, \qquad \sinh q = -\frac{w}{k\mathrm{r}_1}\,. \tag{B.1}$$

Then rotated D_F becomes (we rescale it by an overall factor $\sqrt{\frac{1}{2}(\kappa^2-\nu^2)})$[25]

$$\begin{aligned} D'_F = &\,\Gamma_0\left[\partial_\tau + \frac{\mathrm{r}_0\mathrm{r}_1k^2}{\sqrt{2(\kappa^2-\nu^2)}}\Gamma_{10} - \frac{k\mathrm{w}\mathrm{r}_0\mathrm{r}_1}{\sqrt{2(\kappa^2-\nu^2)}}\Gamma_{13} + \frac{\kappa m}{\mathrm{w}}\frac{\mathrm{w}^2-w^2}{\kappa^2-\nu^2}\Gamma_{16}\right] \\ &- \Gamma_3\left[\partial_\sigma - \frac{\mathrm{r}_0}{\mathrm{r}_1}\frac{mw}{\sqrt{2(\kappa^2-\nu^2)}}\Gamma_{10} - \frac{k^2\mathrm{r}_0\mathrm{r}_1}{\sqrt{2(\kappa^2-\nu^2)}}\Gamma_{13} + \frac{k\kappa m\mathrm{r}_0^2}{\kappa^2-\nu^2}\Gamma_{16}\right] + \\ &+ \frac{2k\kappa\mathrm{r}_0\mathrm{r}_1}{\sqrt{2(\kappa^2-\nu^2)}}\Gamma_{124}\,. \end{aligned} \tag{B.2}$$

To find 16 characteristic frequencies we replace ∂_τ by $i\omega$ and ∂_σ by in and solve the equation $\det D'_F{=}0$ for ω. To simplify the problem may first restrict to the subspaces $\Gamma_{24}\theta = \pm i\theta$ (Γ_{24} commutes with other matrices in D'_F). Then

$$\tilde{D}_F = i\omega\Gamma_0 - i\,n\,\Gamma_3 \pm ia\Gamma_1 + c\Gamma_{016} + d\Gamma_{136}\,, \tag{B.3}$$

where

$$a = \frac{\sqrt{2}k\kappa\mathrm{r}_0\mathrm{r}_1}{\sqrt{\kappa^2-\nu^2}} \qquad c = \frac{\kappa m}{\mathrm{w}}\frac{\mathrm{w}^2-w^2}{\kappa^2-\nu^2}\,, \qquad d = \frac{k\kappa m\mathrm{r}_0^2}{\kappa^2-\nu^2}\,. \tag{B.4}$$

Multiplying (B.3) by Γ_0 we can further restrict to the subspaces $\Gamma_{0136}\theta = \pm i\theta$ (Γ_{0136} commutes with Γ_{24}). The sign degeneracy leads to simple degeneracy in the frequencies and we finish with the following characteristic equation

$$\omega^2 \pm 2d\,\omega - a^2 - c^2 + d^2 \pm 2c\,n - n^2 = 0\,. \tag{B.5}$$

Its solution is

$$\omega = \pm\sqrt{(n\pm c)^2 + a^2}\ \pm d\,. \tag{B.6}$$

There is also an extra factor of 2 degeneracy making up the total 16 of fermionic frequencies in (4.6). One can show that a^2+c^2 is always bigger than d^2 and that $a^2 > d^2$ for $\mathcal{J} > k$ (which is the large $\mathcal{J}$ region we are interested in).

We observe that for any n half of the fermionic frequencies are positive and half negative. According to [49], we expect that in this case $\mathrm{sign}(C^{(n)}_{I,F})\omega^F_{I,n} = |\omega^F_{I,n}|$. As one can check, this is indeed true here. Then taking into account that the square root term in the absolute value of ω in (B.6) is bigger than d,[26] we are led to (4.25).

[25] Some useful relations are $w\cos p - \mathrm{w}\mathrm{r}_1\sin p = \frac{w}{k\mathrm{r}_1}\sqrt{m^2+k^2\mathrm{r}_1^2}$, $m^2+k^2\mathrm{r}_1^2 = \frac{1}{2}(\kappa^2-\nu^2)$, $\kappa\mathrm{r}_0\sinh q + (w\cos p - \omega\mathrm{r}_1\sin p)\cosh q = 0$.

[26] Note that this is true for any $\mathcal{J}$ in the large n region relevant for checking the finiteness of the sum in (4.35) which should hold for any value of $\mathcal{J}$.

C. Landau-Lifshitz frequencies

The characteristic frequencies obtained from the Landau-Lifshitz lagrangian (3.16) should form a subset of all bosonic characteristic frequencies expanded at large $\mathcal{J}$. We can obtain the quadratic fluctuation lagrangian around the (S, J) solution discussed in section 3 by introducing the fluctuations $\eta \to \eta + \tilde{\eta},\ \rho \to \rho_0 + \tilde{\rho}$ and expanding to quadratic order in $\tilde{\eta}, \tilde{\rho}$. We then find that the characteristic frequencies are given by

$$\omega_{\pm} = \frac{1}{2\mathcal{J}} \left[2nk(1 + 2\mathrm{r}_1^2) \pm n\sqrt{n^2 + 4k^2\mathrm{r}_1^2(1 + \mathrm{r}_1^2)} \right], \tag{C.1}$$

These frequencies should be seen from the Bethe ansatz as in [6].

Note, however, that the above frequencies differ from the expansion of the full string frequencies in (4.18) by the term $\frac{nk\mathrm{r}_1^2}{\mathcal{J}}$.[27] The resolution to this puzzle is that one has to match carefully the solution of the Landau-Lifshitz model (obtained from string theory action using uniform gauge) and the string solution obtained in the conformal gauge. Let (τ_u, σ_u) be the world-sheet coordinates of the uniform gauge, and (τ_c, σ_c) the coordinates of the conformal gauge. For the circular solution (3.5) in the conformal gauge one has $\eta = \phi_1 - t = \mathrm{w}\tau_\mathrm{c} + \mathrm{k}\sigma_\mathrm{c} - \kappa\tau_\mathrm{c}$, so that expanding at large $\mathcal{J}$ we get

$$\eta = \frac{k^2}{2\mathcal{J}}\tau_c + k\sigma_c\,. \tag{C.2}$$

In the uniform gauge we have (here we have rescaled (3.16) by $\tau \to \mathcal{J}\tau$ giving $t = \mathcal{J}\tau, \alpha = \mathcal{J}\tau + m\sigma$, as appropriate for comparison to the circular solution in conformal gauge)

$$\eta = \frac{k^2(1 + 2\mathrm{r}_1^2)}{2\mathcal{J}}\tau_u + k\sigma_u\,. \tag{C.3}$$

We see that the above expressions match when the two gauges are related by:

$$\sigma_u = \sigma_c - \frac{k^2\mathrm{r}_1^2}{\mathcal{J}}\tau_c\,, \qquad \tau_u = \tau_c\,. \tag{C.4}$$

Given that the fluctuations are proportional to $e^{i\omega\tau + in\sigma}$ this change would produce the following frequencies in the conformal gauge

$$\omega_{\pm} = \frac{1}{2\mathcal{J}} \left[2nk(1 + \mathrm{r}_1^2) \pm n\sqrt{n^2 + 4k^2\mathrm{r}_1^2(1 + \mathrm{r}_1^2)} \right]. \tag{C.5}$$

These are indeed the same as the frequencies obtained directly by expanding the string theory frequencies in (4.18).

[27] The 1-loop energy computed by summing these frequencies is still the same: the term proportional to n, i.e. $\frac{nk(1+\mathrm{r}_1^2)}{\mathcal{J}}$, does not contribute when taking the sum over n due to the weights $C_I^{(n)}$ with which the frequencies appear in the sum like (4.3).

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